(D-E)-dimensional brane worlds and de Rham distribution formalism: Singular split versus compactification, restrictions on scenario and revision of gravitational energy problem

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Abstract

The proposed so far brane-world cosmological scenarios are concerned with (D-1)-dimensional embeddings into the D-dimensional spacetime, besides, it is supposed D=5 as a rule. However, the regarding of the five-dimensional spacetime as a physical one is a step in past because the modern concepts of superstring theory require to consider our four-Universe as a region inside of a much more higher-dimensional manifold. So, it would be much more realistic to consider our four-Universe as 4-shell or 3-brane inside, e.g., 10-dimensional (or even infinite-dimensional) spacetime. In turn it immediately means that the theory of the $(D-D_E)$-dimensional singular embeddings, where the number of extra dimensions $D_E > 1$, is needed. Hence, the aim of this work is to provide such a theory: we construct the rigorous general theory of the induced gravity on singular submanifolds. At first, we perform the decomposition of the tangent bundle into the two subbundles which will be associated later with external and visible (with respect to some low-dimensional observer) parts of the high-D manifold. Then we go to physics and perform the split of the manifold (in addition to the split of the tangent bundle) to describe both the induced internal geometry and external as-a-whole dynamics of singular embeddings, assuming matter being confined on the singular submanifold but gravity being propagated through the high-D manifold. With the use of the de Rham axiomatic approach to delta-distributions we demonstrate that the four-Universe can be singularly embedded only in five- and six-dimensional space so if we want to consider it's embedding in 10D then extra dimensions must be included as a product space only. We discuss the revealed generic features of the theory such as the multi-normal anisotropy, restrictions on an ambient space, reformulation of the conserved gravitational stress-energy tensor problem, etc.

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I. INTRODUCTION

The theory of pregeometry, i. e., induced rather than imposed gravity, proposed in late 60's by Sakharov [1] (see also the dedicated review [2]) is revived nowadays in the another context - in the higher-dimensional models of the Universe. At early stages of this mainstream the Sakharov's idea was applied to n-dimensional manifolds [3], as a rule in the connection with the Kaluza-Klein paradigm (i.e., assuming the further compactification of extra dimensions). At the same time there appeared the "Universe as vortex" model [4] which demonstrated that the KK compactification is not the only way of the "hiding" of extra dimensions but that idea remained almost unnoticed on the background of the universal popularity of KK-type theories.

Much later, being inspired by the two works of Rubakov and Shaposhnikov [5] (the first paper was devoted to the "Universe as domain wall" conjecture whereas the second one proposed a high-dimensional solution of the hierarchy problem), Gogberashvili [6] and later independently of him Randall and Sundrum [7] have put forward the (4+1)-dimensional brane-world cosmological scenarios with the emphasis on the hierarchy problem. Despite the difference of approaches (Gogberashvili used the geometrical junction theory [8–11] whereas Randall and Sundrum did the variational method with the stress-energy tensor containing delta-functions), terminology (Gogberashvili called it as "Universe as shell" scenario whereas Randall and Sundrum used the modern "Universe as three-brane") and physical assumptions, all three authors, in fact, proposed the same point of view which amplified the interest [12] to the high-dimensional cosmological models where extra dimensions were assumed to be orthogonal to the Universe as a singular shell or 3-brane rather than compactified.

The further research efforts were directed toward the diversifying of the physically specific models of five-dimensional brane-world cosmology as well as toward the elucidation of the relations between the two above-mentioned approaches [13,15,17] (the class of branes belongs to much more wide family of singular shells; the integration of Einstein equations with the distributional sources can be reformulated in terms of the more rigorous junction formalism [16], etc.), the generalizing in several aspects [17], and the considering of high-dimensional brane models, e.g., $D8$-branes (Dirichlet 9-embeddings) in a 10-dimensional SUGRA spacetime [13].

Overlooking the geometrical achievements of the "brane-world rush" one can reveal that all the studies are concerned with $(D-1)$-dimensional embeddings and their special case, $(D-2)$-branes, into the D-dimensional spacetime, besides it is supposed $D = 5$ as a rule. However, the serious considering of five-dimensional spacetime means, in fact, step back in past because the modern concepts of superstring theory require to consider our four-Universe as a region inside of a much more high dimensional spacetime. In turn, it immediately means that we are needed in the theory of the geometrically induced gravity on $(D - D_E)$-dimensional embedded (singular) manifolds where the number of extra dimensions $D_E > 1$. To the best of our knowledge such a singular junction formalism has not yet been presented anywhere despite the embedding of Riemann surfaces into a higher-dimensional spacetime is a well-studied classical problem [14]. The aim of this work is thus to equip a reader with it.

The emphasis will be done on the fundamental aspects of the theory because the early falling into physical particularities can raise some confusing whether discussed properties are generic or not. Thus, we will try to construct the (more or less) rigorous general formalism of the singular submanifold theory founding on the geometrical junction theory because the straightforward integration of Einstein equations with distributions is good for quick obtaining of certain results rather than for full understanding of what we are doing. Throughout the

paper we will emphasize on the difference between geometries of the Kaluza-Klein type models and the theory of singular submanifolds.

In Sec. II we perform the decomposition of the tangent bundle into the two subbundles which will be associated later with external and visible (with respect to some low-dimensional observer) parts of the high-D manifold. This section is common for both the KK models and the singular submanifold theory, and provides us with underlying mathematical language. In Sec. III we go to physics and perform the split of the manifold (in addition to the split of the tangent bundle) to describe the induced internal geometry and external as-a-whole dynamics of singular embeddings. We assume matter (including the Standard Model with its fiber bundles) to be confined on the singular (sub)manifold and introduce the multi-normal surface stress-energy tensor. We consider both the direct (naive) approach and the axiomatic theory of delta-like distributions based on the de Rham currents. In Sec. IV we discuss the features of the brane world viewpoint in comparison with those of the KK models.

II. $(V+E)$-DECOMPOSITION OF TANGENT BUNDLE

Let us consider the D-dimensional Riemann manifold Σ assuming T_Σ is its underlying tangent bundle space. Let us cover the bundle by D basis vectors $\mathbf{e}_\alpha$ (here and below Greek indices run from 1 to D). Further, let us divide the set $\{\mathbf{e}_\alpha,\ \alpha = 1,\ 2, ..., D\}$ into the two subsets $\{\mathbf{e}_i,\ i = 1,\ 2, ..., D_V\}$ and $\{\mathbf{e}_a,\ a = D_V+1, ..., D_V+D_E\}$ where $D_E+D_V = D$. Further it will be everywhere assumed that Latin indices i, j, k, l, m run from 1 to D_V, a, b, c, d, f do from D_V+1 to $D_V+D_E = D$.

Then, if there are imposed some mathematical rules for the dividing the set $\{\mathbf{e}_\alpha\}$ into the sum $\{\mathbf{e}_i\} \cup \{\mathbf{e}_a\}$ it means that T_Σ is decomposed into the two subbundles which we will call as $T_{E(xtra)}$ and $T_{V(isible)}$ keeping in mind the forthcoming physics which will be based on this formalism. In reality, it is enough to restrict ourselves by the case $D_V = 4$ but in this paper we will study the most general case of arbitrary D_E and D_V.

Note, the decomposition $T_\Sigma = T_E \oplus T_V$ does not mean yet the split of the D-dimensional manifold Σ into the sum of the (singular) submanifolds E and V. At this stage we just have $(V+E)$-relabeled the underlying bundle space of Σ to obtain some useful basic formulae which in their turn will gain a concrete physical sense *only* when, running ahead, considering the related physical entities, singular submanifolds.

The $(V+E)$-decomposition of tangent bundle space is the natural generalization of the $(V+1)$-decomposition, the basic formalism of $(D-1)$-embeddings in D-spacetime, on the case $D_E > 1$. The $(V+1)$-decomposition (especially its special cases $3+1$ and $4+1$) happened to be excellent language for singular shell theory, Hamiltonian formulation of general relativity and Cauchy problem in GR, but, as was mentioned above, the modern concepts demand for the language for the description of more "compact" $(D_V < D-1)$ embeddings into high-D spacetime.

Further, for simplicity we will suppose the basis $\{\mathbf{e}_\alpha\} = \{\{\mathbf{e}_i\}, \{\mathbf{e}_a\}\}$ to be orthogonal and commutative,

$$\mathbf{e}_a \cdot \mathbf{e}^b = \delta_a^b,\ \mathbf{e}_i \cdot \mathbf{e}^k = \delta_i^k, \tag{1}$$

besides we will assume the block-orthogonality condition

$$\mathbf{e}_a \cdot \mathbf{e}_i = 0. \tag{2}$$

Assuming that the connection is symmetric and compatible with metric we can decompose it into $V(isible)$ and $E(xtra)$ parts as well:

$$\nabla_{\mathbf{e}_\nu}\mathbf{e}_\mu = K^a{}_{\nu\mu}\mathbf{e}_a + K^i{}_{\nu\mu}\mathbf{e}_i, \quad K_{\alpha\nu\mu} \equiv \mathbf{e}_\alpha \cdot \nabla_{\mathbf{e}_\nu}\mathbf{e}_\mu, \tag{3}$$

and one can check that connections with non-mixed "E, V"-indices coincide with the Christoffel symbols in the corresponding subbundle V or E:

$$K^a_{\ bc} = {}^{(E)}\Gamma^a{}_{bc}, \quad K^i{}_{jk} = {}^{(V)}\Gamma^i{}_{jk}, \tag{4}$$

so that below when dealing with Christoffel symbols we will omit the superscripts $^{(V)}$ and $^{(E)}$ for brevity. With this in hands we can $(V+E)$-decompose all the necessary tensors. For some needed components of the Riemann tensor in natural frame we hence have

$$R^i{}_{jkl} = {}^{(V)}R^i{}_{jkl} + K^i{}_{a[k}K^a{}_{l]j}, \tag{5}$$

$$R^a{}_{jkl} = K^a{}_{j[l;k]} + K^a{}_{\lambda[l}K^\lambda{}_{k]j}. \tag{6}$$

These expressions are the generalizations of the Gauss-Codacci equations of the $(V+1)$-decomposition which in turn is the underlying formalism both for the singular shell theory in the ordinary spacetime $D=4$ [11] and for the proposed brane-world (toy) models of the four-Universe as a 3-brane in the higher-dimensional space with $D=5$ [15,17]. Indeed, if the E-index a has only one value, $a_{D_V+1}=n$, we obtain

$${}^{(D)}R^m_{\ ijk} = {}^{(D-1)}R^m_{\ ijk} + g^{nn}K_{i[j}K^m_{k]}, \tag{7}$$

$${}^{(D)}R^n_{\ ijk} = g^{nn}K_{i[k;j]}, \tag{8}$$

where the extrinsic curvature $K_{ij} \equiv K_{nij}$, and certain features of the gaussian/synchronous reference frame were taken into account. Note, that the equation (6) in comparison with eq. (8) contains the extra (second) term which is caused by the fact $D_E > 1$. Running ahead, we say that appearance of such terms is inevitable and sufficiently complicates matter.

The components of the decomposed Einstein tensor are

$$G^i_k = {}^{(V)}G^i_k + R^{ai}{}_{ak} - g^i_k R^{aj}{}_{aj} - \frac{1}{2}g^i_k\left({}^{(E)}R + K^l{}_{a[j}K^{aj}{}_{l]} + K^a{}_{j[c}K^{jc}{}_{a]}\right) + K^j{}_{a[k}K^{ai}{}_{j]}\,, \tag{9}$$

$$G^i_d = R^i_d = K^{ai}{}_{[d,a]} + K^{ji}{}_{[d,j]} + K^i{}_{\lambda[a}K^{a\lambda}{}_{d]} + K^i{}_{\lambda[j}K^{j\lambda}{}_{d]}\,, \tag{10}$$

$$G^c_d = {}^{(E)}G^c_d + R^{cj}{}_{dj} - g^c_d R^{aj}{}_{aj} - \frac{1}{2}g^c_d\left({}^{(V)}R + K^a{}_{j[b}K^{jb}{}_{a]} + K^l{}_{a[j}K^{aj}{}_{l]}\right) + K^a{}_{j[d}K^{jc}{}_{a]}\,, \tag{11}$$

where we defined

$$R^{ai}{}_{bk} = K^{ai}{}_{[k,b]} + K^i{}_{\lambda[b}K^{a\lambda}{}_{k]}\,.$$

Now we have all the necessary formulae to consider the geometry and physics of D_V-dimensional singular embeddings.

III. SINGULAR SPLIT: GEOMETRY ENCOUNTERS PHYSICS

So far the $(V+E)$-decomposition formulae just represented the split of the tangent bundle hence were nothing but the simple relabeling of the base manifold Σ. Now let us suppose that

there exists some entity that performs not only the split of the bundle $T_\Sigma = T_E \oplus T_V$ but also the split of the base manifold into the parts $\Sigma = E(xtra) \cup V(isible)$.[1] Each of these two singular submanifolds can be assumed to have its own geometry and matter on its D_E- or D_V-dimensional worldsheet. Besides the intrinsic geometry a singular submanifold can move as a whole inside the parent spacetime Σ hence it has own non-trivial external dynamics. Unlike this, by definition the non-singular manifold has neither (hyper)surface matter nor external dynamics, and represents itself just some (relative) region of Σ having no physical carrier.

For definiteness, we select for further studying the singular submanifold V assuming E as the rest, $E = \Sigma/V$. The embedding V will be associated with our visible four-dimensional Universe hence $D_V = 4$ but for generality we will assume arbitrary $D_V < D$.

Thus, the singular submanifold V appears to be the physical carrier that "fixes" the $(V+E)$-decomposition. The question now is how to define the intrinsic stress-energy tensor of the matter on its (hyper)surface. For the $(D-1)$-dimensional singular embeddings (the standard thin-shell formalism) we had the following definition of surface stress-energy tensor

$$
{}^{(V+1)}S^\alpha_\beta = \lim_{\varepsilon \to 0} \int\limits_{-\varepsilon}^{\varepsilon} T^\alpha_\beta \mathrm{d}n, \tag{12}
$$

where T^α_β is the general D-dimensional stress-energy tensor, n is the (only) extra coordinate "piercing" the submanifold. However, now our submanifold V has D_E normals towards the E(xtra) directions $\{\mathbf{n}^a\}$ $(a = D_V + 1, ..., D)$ so the question now is how to generalize the standard thin-shell concepts.

A. Direct approach reveals contradictions

The most natural generalization of the integral in eq. (12) seems to be the integral $\int T^\alpha_\beta \mathrm{d}n^a$ but it immediately does mean the appearance of an extra index at S^α_β (this is required also by the left-hand sides of eqs. (18) - (20)). Then the genuine surface stress-energy tensor is given by the following sum

$$
S^\alpha_\beta = \prod_{a=D_V+1}^{D=D_V+D_E} S^{(a)\alpha}{}_\beta, \quad S^{(a)\alpha}{}_\beta = \lim_{\varepsilon \to 0} \int\limits_{-\varepsilon}^{\varepsilon} T^\alpha_\beta \, n^a_\mu \mathrm{d}x^\mu, \tag{13}
$$

where $\mathbf{n}^a$'s are $D - D_V = D_E$ normal vectors to V. From the viewpoint of the observer living inside V this index seems to be numbering internal (non-spacetime) degrees of freedom because its lowering/raising is governed by the E-metric g_{ab} only, by virtue of the block-orthogonality condition. In fact, eq. (13) reflects itself the new mechanism of generation of internal degrees of freedom which will be referred throughout the paper as the *multi-normal anisotropy* and discussed in more details later.

Assuming that the Einstein equations in D-spacetime Σ,

$$
G^\mu_\nu = k_D^2 T^\mu_\nu, \tag{14}
$$

[1]Throughout the paper we will call the former as the *decomposition* or *simple split* and the latter as the *singular split*.

are valid, we obtain from eq. (9) - (11) both the equation for induced gravity on V,

$$ {}^{(V)}G^i_k = k_D^2 T^i_k - R^{ai}{}_{ak} + g^i_k R^{aj}{}_{aj} + \frac{1}{2} g^i_k \left({}^{(E)}R + K^l{}_{a[j} K^{aj}{}_{l]} + K^a{}_{j[c} K^{jc}{}_{a]} \right) - K^j{}_{a[k} K^{ai}{}_{j]} \,, \tag{15} $$

where all R's are supposed to be the functions of the D-dimensional stress-energy tensor, and the additional equations

$$ K^{ai}{}_{[d,a]} + K^{ji}{}_{[d,j]} + K^i{}_{\lambda[a} K^{a\lambda}{}_{d]} + K^i{}_{\lambda[j} K^{j\lambda}{}_{d]} = k_D^2 T^i_d \,, \tag{16} $$

$$ {}^{(E)}G^c_d + R^{cj}{}_{dj} - g^c_d R^{aj}{}_{aj} - \frac{1}{2} g^c_d \left({}^{(V)}R + K^a{}_{j[b} K^{jb}{}_{a]} + K^l{}_{a[j} K^{aj}{}_{l]} \right) + K^a{}_{j[d} K^{jc}{}_{a]} = k_D^2 T^c_d \,, \tag{17} $$

which will be also important below.

Further, applying the pill-box integration over all E-coordinates (with successive taking of the limit $\varepsilon \to 0$) to the Einstein equations (14) in the decomposed form (15)-(17) and assuming that Σ-metric is continuous across V we obtain that the first derivatives have a finite jump across V, and the comparison of the integrands at $\int d^{E-1}x^a$ (which is, by definition, the integration over all E-coordinates except ath) yields the junction conditions

$$ [K]^{ai}{}_k - \delta^i_k [K]^{aj}{}_j - \frac{1}{2} \delta^i_k [\Gamma]^{[ab]}{}_b = \frac{k_D^2}{D_E} S^{(a)i}{}_k \,, \tag{18} $$

$$ [K]^{ai}{}_d - \delta^a_d [K]^{bi}{}_b - \delta^a_d [\Gamma]^{ji}{}_j = \frac{k_D^2}{D_E} S^{(a)i}{}_d \,, \tag{19} $$

$$ [\Gamma]^{ac}{}_d - \delta^a_d [\Gamma]^{bc}{}_b - \frac{1}{2} \delta^a_d [\Gamma]^{[ab]}{}_b + \delta^{[a}_d [K]^{c]i}{}_i = \frac{k_D^2}{D_E} S^{(a)c}{}_d \,, \tag{20} $$

where it was used that only the derivatives with respect to x^a survive, and the jump "[]" is defined as

$$ [Z] = Z(x^i, n^a_\mu x^\mu = n^a_\mu x^\mu_0 + 0) - Z(x^i, n^a_\mu x^\mu = n^a_\mu x^\mu_0 - 0), $$

where the coordinates x^μ_0 point out the position of V. Considering eq. (13), the Σ-stress-energy tensor can be imagined in the split form as the superposition of the B(ulk) part and the V-surface's part as the sum of S's over all D_E normals

$$ T^\mu_\nu = B^\mu_\nu + \prod_a S^{(a)\mu}{}_\nu \, \delta(n^a_\mu x^\mu - n^a_\mu x^\mu_0), \tag{21} $$

where the second term is zero everywhere except V.

The system of eqs. (14) - (21) completely determines both the intrinsic geometry of V and the external dynamics of V as a whole inside the parent manifold. In some cases it can be greatly simplified, e.g., if one assumes E-coordinates to be flat (i.e., gaussian/synchronous: $g_{ab} = \text{const}$ hence $\Gamma^a{}_{bc} = 0$, $K_{aib} = 0$, $K_{aij} = -(1/2) g_{ij,a}$, etc.), and/or if D-dimensional manifold admits Z_2-symmetry $[Z] = \pm 2Z$.

Everything looks fine but in practice it is not so. The problem is that the stress-energy tensor given by eqs. (13) and (21) does not describe the desired V-embedding. It describes instead the aggregate of the orthogonal singular hyperplanes crossing the V submanifold. Moreover, it seems impossible to reconcile the definition of stress-energy tensor based on the pill-box integration with the circumstance that the L.H.S. (hence R.H.S.) of eqs. (21) acquire an extra E-index a if $D_E > 1$. The reasons of this contradiction happen to be quite deep - we have to consider the axiomatic theory of delta-like distributions to clear them up.

B. Axiomatic approach to singular embeddings: de Rham currents

Let us start from the very initial definitions - the definitions of delta-like distributions within the frameworks of the geometrical theory of distributions [19] (we will use also Ref. [20] where that theory was adapted to thin-shell formalism).

Let $C_0^\infty(R^n)$ be the vector space of compactly supported smooth functions on R^n. If $(x^1, ..., x^n)$ are the canonical coordinates on R^n, we define the operators $D_i = \partial/\partial x^i$, $D^\alpha = D^{\alpha_1}...D^{\alpha_n}$, with $\alpha = \{\alpha_1, ..., \alpha_n\}$ to be integer non-negative numbers. The C^p - topology is defined on $C_0^\infty(R^n)$ by saying that the sequence φ_n tends to zero if: there is a compact set K with $\mathrm{supp}(\varphi_n) \subset K$ and $D^\alpha\varphi_n$ vanishes uniformly for $x \in K$ and all α satisfying $|\alpha| = \alpha_1 + ... + \alpha_n \leq p$.

Then the *distribution* on R^n is the linear map T: $C_0^\infty(R^n) \mapsto R$, and it is continuous in the C^∞ topology. The vector space of distributions on R^n is denoted $D(R^n)$. Now we extend the operators D_i to the space of distributions by setting

$$(D_i T, \varphi) = (-1)^{|\alpha|}(T, D_i\varphi),\ \forall\varphi \in C_0^\infty(R^n),$$

and denote by $\Omega^*(U)$ the algebra of exterior forms on U where U is a domain of R^n. Let $\Omega_c^q(R^n)$ be the space of q-forms on R^n with compact support recalling that the q-form with compact support is $\omega = \sum \omega_{i_1...i_q} \mathrm{d}x^{i_1}...\mathrm{d}x^{i_q}$ where $\omega_{i_1...i_q} \in C_0^\infty(R^n)$, $1 \leq i_1 \leq ... \leq i_q \leq n$.

The topological dual of $\Omega^{n-q}c(R^n)$ is the space of the (de Rham) currents of degree q and is denoted by $D^q(R^n)$, then the current $T \in D^q(R^n)$ may be considered as a differential form $T = \sum \omega_I \mathrm{d}x^I$, $|I| = q$, where $I = i_1, ..., i_q$ and distribution coefficients ω_I are defined by $(\omega_I, \varphi) = \pm(T, \varphi \mathrm{d}x^{I^0})$, $\forall\varphi \in C_0^\infty(M)$, where I^0 is the compound index defined by $\star\mathrm{d}x^I = \pm\mathrm{d}x^{I^0}$. The exterior derivative on smooth forms induces an exterior derivative operator on the spaces of currents:

$$\mathrm{d} : D^q(R^n) \mapsto D^{q+1}(R^n),\quad (\mathrm{d}T, \varphi) = (-1)^q(T, \mathrm{d}\varphi).$$

To see explicitly the links between this formalism and usual definition of delta-like distributions let us localize the definitions of the de Rham currents. Let M be a smooth manifold of dimension n. If U is an open set in M the space of distributions with support in U, $D(U)$, is the topological dual of the space $C_0^\infty(U)$. Let $\Omega_0^*(M)$ denote the algebra of compactly supported q-forms on M and $\Omega_0^*(U)$ do the space of smooth compactly supported q-forms on U. Taking the topological duals of these spaces we obtain the space of distributions $D(M)$ and the spaces of currents $D^q(M)$ and $D^q(U)$. For now on we will consider only space-time manifolds M. Let S be the hypersurface in M defined by the equation $\eta(x^\alpha) = 0$, $\nabla\eta \neq 0$, where x^α are local coordinates on M. Denote by $\theta(\eta)$ the characteristic function of the n-dimensional domain $\Gamma = \{\eta > 0\}$ with the boundary $\Sigma = \partial\Gamma$

$$\theta(\eta) = \begin{cases} 1 & \eta \geq 0 \\ 0 & \eta < 0 \end{cases},$$

so $\theta(\eta)$ defines a current by

$$(T_\theta, \varphi) = \int \theta \wedge \varphi = \int\limits_{\eta \geq 0} \varphi,$$

hence

$$(T_{\mathrm{d}\theta}, \varphi) = \int \mathrm{d}\theta \wedge \varphi = -\int\limits_{\Gamma} \mathrm{d}\varphi,$$

and one can prove that we can uniquely associate with Σ a closed 1-form $\delta(\Sigma) \in H^1(M)$ (with $H^q(M)$ to be the q-th de Rham cohomology of M) so that $\mathrm{d}\theta(\eta) = \delta(\eta)\mathrm{d}\eta$ which denotes that $D_i\theta(\eta) = D_i\eta\delta(\eta)$. Thus, the delta "function" $\delta(\Sigma)$ has been defined as the special exact 1-form.

Finally, let us give the integral representation of the distribution $\delta(\Sigma)$. Let $\chi(x) \in C_0^\infty(M)$ be the nonnegative function on M, supported in the vicinity of Σ with $\int \chi(x)dx = 1$. We set $\chi_\varepsilon(x) = \varepsilon^{-n}\chi(x/\varepsilon)$. Now if $\mathrm{supp}\chi = K$ then $\mathrm{supp}\chi_\varepsilon = K_\varepsilon$ and $\int \chi_\varepsilon(x)dx = 1$. Then $T_{\chi_\varepsilon} \to \delta$ as $\varepsilon \to 0$ in the sense that

$$\lim_{\varepsilon\to 0}(T_{\chi_\varepsilon}, \varphi) = (\delta, \varphi), \;\; \forall\varphi \in C_0^\infty(M),$$

i.e. we can approximate the delta-singularities by smooth functions in this way. Now we have all the necessary definitions to consider the thin-shell (singular embedding) theory within the frameworks of de Rham currents' approach.

Standard ($D_E = 1$) thin-shell theory and de Rham currents

Again, let M be a space-time manifold and M_+ and M_- be two overlapping domains of this manifold, Σ be the hypersurface contained in $M_+ \cap M_-$ which embedding is defined in local coordinates by the equation $\eta(x^\alpha) = 0$. For the pair of metrics $g_{ab}^{\pm}$ defined on $M^{\pm}$ respectively, assuming coordinates be continuous across Σ, we will impose the condition

$$[g_{\alpha\beta}] \equiv g_{ab}^{+} - g_{ab}^{-} \mid_\Sigma = 0. \tag{22}$$

Also we have

$$g_{\pm}^{\alpha\beta}\, \partial_\alpha\eta\partial_\beta\eta = \varepsilon_{\pm}(\eta)\alpha_{\pm}^2(x),$$

so

$$n_\alpha^{\pm} = \frac{1}{\alpha_\pm}\partial_\alpha\eta,$$
$$g_{\pm}^{\alpha\beta} n_\alpha^{\pm} n_\beta^{\pm}|_\Sigma = \varepsilon_\pm(x) = \begin{cases} 0, & \Sigma \text{ is lightlike} \\ \pm 1, & \text{otherwise} \end{cases} .$$

To describe uniformly both null and non-null surfaces let us introduce the vector N_α such that

$$N^\alpha n_\alpha^{\pm} = 1/\zeta_\pm, \quad \text{i.e.,} \quad N^\alpha \partial_\alpha \eta = \alpha/\zeta_\pm,$$

where ζ's are some functions, for definiteness ζ is 1 if Σ is timelike and -1 otherwise. This allows us to consider the case of the null surface Σ as the limit $\varepsilon \to 0$ of the non-null one. In terms of the vielbein ω^α the metric in M is

$$ds^2 = \eta_{\alpha\beta}\omega^\alpha\omega^\beta,$$

and in presence of Σ we have

$$\omega^\alpha = \omega_+^\alpha\theta(\eta) + \omega_-^\alpha\theta(-\eta), \tag{23}$$

where θ is the 0-form current defined above as the (analogue of the) Heaviside function. The condition (22) implies

$$(\omega_+^\alpha - \omega_-^\alpha)|_\Sigma = 0,$$

hence we have $\mathrm{d}\omega^\alpha = \mathrm{d}\omega_+^\alpha \theta(\eta) + \mathrm{d}\omega_-^\alpha \theta(-\eta)$, and the connection 1-form is given by the first Cartan equation $\mathrm{d}\omega^\alpha = \omega^\beta \wedge \Gamma^\alpha_\beta$ as

$$\Gamma^\alpha_\beta = \Gamma^{\alpha+}_\beta \theta(\eta) + \Gamma^{\alpha-}_\beta \theta(-\eta). \tag{24}$$

The second Cartan equation yields the curvature 2-form decomposed into two bulk ($\pm$) parts and one singular on Σ

$$R^\alpha_\beta = \mathrm{d}\Gamma^\alpha_\beta + \Gamma^\alpha_\gamma \wedge \Gamma^\gamma_\beta = R^{\alpha+}_\beta \theta(\eta) + R^{\alpha-}_\beta \theta(-\eta) + \delta(\eta) S^\alpha_\beta, \tag{25}$$

where we have defined

$$S^\alpha_\beta = \mathrm{d}\eta \wedge [\Gamma^\alpha_{\beta\gamma}]\omega^\gamma,$$

and $[\Gamma^\alpha_{\beta\gamma}]\omega^\gamma \equiv (\Gamma^{\alpha+}_\beta - \Gamma^{\alpha-}_\beta)|_\Sigma$, and it was used the definition $\mathrm{d}\theta = \delta \mathrm{d}\eta$ from the introductory part above. Further, using $\mathrm{d}\eta = \alpha n_\gamma \omega^\gamma$ we obtain

$$S^\alpha_{\beta\gamma\sigma} \omega^\gamma \wedge \omega^\sigma = -2\alpha [\Gamma^\alpha_{\beta\gamma}] n_\sigma \omega^\gamma \wedge \omega^\sigma,$$

or, simply,

$$S^\alpha_{\beta\gamma\sigma} = -2\alpha [\Gamma^\alpha_{\beta\gamma}] n_\sigma. \tag{26}$$

Let us imply now the coordinate basis. Then the jump of the first derivative of metric across Σ is

$$[\partial_\mu g_{\alpha\beta}] = \zeta \gamma_{\alpha\beta} n_\mu,$$

where $\gamma_{\alpha\beta}$ is the jump in the transversal derivative $\gamma_{\alpha\beta} = \alpha N^\mu [\partial_\mu g_{\alpha\beta}]$, so that the jump of the Christoffel symbols across Σ can be expressed as

$$[\Gamma^\alpha_{\beta\gamma}] = \zeta (\gamma^\alpha_\beta n_\sigma + \gamma^\alpha_\sigma n_\beta - \gamma_{\beta\sigma} n_\alpha)/2, \tag{27}$$

so the surface Riemann and Ricci tensors are, respectively,

$$S^\alpha_{\beta\gamma\delta} = \frac{\alpha}{2}\zeta \left[n^\alpha (\gamma_{\beta\delta} n_\gamma - \gamma_{\beta\gamma} n_\delta) - n_\beta (\gamma^\alpha_\gamma n_\delta - \gamma^\alpha_\delta n_\gamma) \right], \tag{28}$$

$$S_{\alpha\beta} = \frac{\alpha}{2}\zeta \left[\gamma_\alpha n_\beta + \gamma_\beta n_\alpha - \gamma n_\alpha n_\beta - \breve{\gamma} h_{\alpha\beta} - \varepsilon(\gamma_{\alpha\beta} - \gamma h_{\alpha\beta}) \right], \tag{29}$$

where $\gamma^\alpha = \gamma^\alpha_\beta n^\beta$, $\breve{\gamma} = \gamma^\alpha n_\alpha$, $\gamma = \gamma_{\alpha\beta} h^{\alpha\beta}$, $h_{\alpha\beta}$ is the metric on Σ. As was promised we are able to obtain the surface Ricci tensor for light-like shells by taking the $\varepsilon \to 0$ limit:

$$S_{\alpha\beta} = \frac{\eta}{8\pi} \left(\gamma_\alpha n_\beta + \gamma_\beta n_\alpha - \gamma n_\alpha n_\beta - \breve{\gamma} h_{\alpha\beta} \right), \tag{30}$$

i.e., the results of ref. [11] were completely reproduced.

Embeddings with $D_E > 1$ in light of de Rham approach
So far we assumed $D_E = 1$, i.e., (D-1)-dimensional layer in D-dimensional manifold. Let us turn now to the case of Σ with $D_E > 1$ which embedding is described by D_E equations

$$\eta^{(a)}(x) = 0, \ a = 1, .., D_E.$$

The most unexpected thing which appears is that D_E *cannot be more than two*! Indeed, within frameworks of the de Rham approach the N-dimensional delta-singularity must be described by the N-form $\mathrm{d}\theta(\eta^{(1)}) \wedge \mathrm{d}\theta(\eta^{(2)}) \wedge .. \wedge \mathrm{d}\theta(\eta^{(N)}) = \delta(\eta^{(1)})..\delta(\eta^{(N)}) \, \mathrm{d}\eta^{(1)} \wedge .. \wedge \mathrm{d}\eta^{(N)}$. This must appear in the singular part of the curvature 2-form (25). But the latter is a 2-form therefore singular part must be a 2-form as well, and one cannot insert there the 3-form $\delta(\eta^{(1)})\delta(\eta^{(2)})\delta(\eta^{(3)}) \, \mathrm{d}\eta^{(1)} \wedge \mathrm{d}\eta^{(2)} \wedge \mathrm{d}\eta^{(3)}$ or higher.

All this does not mean however that we cannot embed the object having the dimension $D - 3$, $D - 4$, etc. into a D-dimensional manifold - simply in that case the object "sees" (at most) two dimensions whereas others form a product space *a la* Kaluza-Klein. For example, a point particle (zero-dimensional object) can be embedded into a four-dimensional spacetime but the Schwarzschild metric is a product space of S^2 and two-dimensional time-radius part hence one can say that a point particle "sees" two dimensions.

Further, the case $D_E = 2$ deserves for special treatment because it is a limit case besides it comprises two-dimensional strings in four-spacetime or four-vertices in six-dimensional spacetime (the latters also were used in the early brane-world proposals [4]). If $D_E = 2$ then the surface curvature 2-form in eq. (25) is some 0-form times the 2-form $\mathrm{d}\eta^{(1)} \wedge \mathrm{d}\eta^{(2)}$. Further, if we want to obtain this from somehow decomposed vielbein the problem is how to find this "somehow". The case $D_E = 1$ is sharply distinct because there Σ is (D-1)-dimensional and hence one is able to introduce the notions "on one side of Σ" and "on another side of Σ" but they are meaningless when $D_E > 1$ so we cannot start with something like eq. (23).

So, let assume that $D_E = 2$ and the embedding of (D-2)-dimensional Σ is described by two equations

$$\eta^{(a)}(x) = 0, \ a = 1, 2,$$

i.e., Σ is the intersection of the two (D-1)-dimensional surfaces, $\mathcal{B}_1$ and $\mathcal{B}_2$, described by the equations $\eta^{(1)}(x) = 0$ and $\eta^{(2)}(x) = 0$ respectively. Instead of eq. (23) we assume

$$\omega^\alpha = \omega^\alpha_{++}\theta_1\theta_2 + \omega^\alpha_{+-}\theta_1\theta_{-2} + \omega^\alpha_{-+}\theta_{-1}\theta_2 + \omega^\alpha_{--}\theta_{-1}\theta_{-2}, \tag{31}$$

where it is denoted $\theta_{\pm a} \equiv \theta(\pm\eta^{(a)})$ for brevity, in hope that later on we will find the restrictions for these ω's because for unrestricted ω's this equation describes the two above-mentioned intersecting surfaces whereas we are interested in their intersection region (Σ) only.

After taking external derivative we obtain

$$\begin{aligned}\mathrm{d}\omega^\alpha = \mathrm{d}\omega^\alpha_{++}\theta_1\theta_2 + \mathrm{d}\omega^\alpha_{+-}\theta_1\theta_{-2} + \mathrm{d}\omega^\alpha_{-+}\theta_{-1}\theta_2 + \mathrm{d}\omega^\alpha_{--}\theta_{-1}\theta_{-2} + \\ (\Delta^\alpha_2\theta_2 + \tilde{\Delta}^\alpha_2\theta_{-2}) \wedge \mathrm{d}\theta_1 + (\Delta^\alpha_1\theta_1 + \tilde{\Delta}^\alpha_1\theta_{-1}) \wedge \mathrm{d}\theta_2,\end{aligned} \tag{32}$$

where

$$\begin{aligned}\Delta_1 \equiv (\omega_{++} - \omega_{+-})\,|_{\mathcal{B}_2}, \ \tilde{\Delta}_1 \equiv (\omega_{-+} - \omega_{--})\,|_{\mathcal{B}_2}, \\ \Delta_2 \equiv (\omega_{++} - \omega_{-+})\,|_{\mathcal{B}_1}, \ \tilde{\Delta}_2 \equiv (\omega_{+-} - \omega_{--})\,|_{\mathcal{B}_1},\end{aligned} \tag{33}$$

and, of course, it should not be forgotten that $\mathrm{d}\theta_a = \delta(\eta^{(a)})\mathrm{d}\eta^{(a)}$. Then after tedious but straightforward calculation we obtain that the analogue of eq. (24) is

$$\Gamma^\alpha_\beta = \Gamma^{++}{}^\alpha_\beta\theta_1\theta_2 + \Gamma^{+-}{}^\alpha_\beta\theta_1\theta_{-2} + \Gamma^{-+}{}^\alpha_\beta\theta_{-1}\theta_2 + \Gamma^{--}{}^\alpha_\beta\theta_{-1}\theta_{-2} +$$
$$(\Gamma_{(2)}{}^\alpha_\beta\theta_2 + \tilde\Gamma_{(2)}{}^\alpha_\beta\theta_{-2})\,\mathrm{d}\theta_1 + (\Gamma_{(1)}{}^\alpha_\beta\theta_1 + \tilde\Gamma_{(1)}{}^\alpha_\beta\theta_{-1})\,\mathrm{d}\theta_2, \tag{34}$$

where $\Gamma^{\pm\pm}{}^\alpha_\beta$ are the standard connection 1-forms calculated from the corresponding $\omega_{\pm\pm}$, and the numbered 0-forms $\Gamma_{(1,2)}{}^\alpha_\beta$ are the solutions of the following linear equations

$$\theta(0)\sigma^\beta_a\Gamma_{(a)}{}^\alpha_\beta = \Delta^\alpha_a, \quad \theta(0)\tilde\sigma^\beta_a\tilde\Gamma_{(a)}{}^\alpha_\beta = \tilde\Delta^\alpha_a, \quad (a=1,2), \tag{35}$$

where it has been defined

$$\sigma_1 \equiv (\omega_{++} + \omega_{+-})\,|_{\mathcal{B}_2}, \; \tilde\sigma_1 \equiv (\omega_{-+} + \omega_{--})\,|_{\mathcal{B}_2},$$
$$\sigma_2 \equiv (\omega_{++} + \omega_{-+})\,|_{\mathcal{B}_1}, \; \tilde\sigma_2 \equiv (\omega_{+-} + \omega_{--})\,|_{\mathcal{B}_1}, \tag{36}$$

besides we have not specified the value of the Heaviside 0-form when its argument is zero.

To find the decomposed curvature 2-form we have to take again the external derivative of Γ^α_β and eventually we obtain that the curvature form consists of the following three parts - the bulk part which does not contain delta-singularities, the first brane part which describes the hypersurface $\mathcal{B}_1$ and is proportional to $\mathrm{d}\theta_1$, the second brane part which describes the hypersurface $\mathcal{B}_2$ and is proportional to $\mathrm{d}\theta_2$, and the intersection part which describes $\Sigma = \mathcal{B}_1\cap\mathcal{B}_2$ and is a certain 0-form times the two-dimensional delta-"function" $\mathrm{d}\theta_1\wedge\mathrm{d}\theta_2$:

$$R^\alpha_\beta = R^{++}{}^\alpha_\beta\theta_1\theta_2 + R^{+-}{}^\alpha_\beta\theta_1\theta_{-2} + R^{-+}{}^\alpha_\beta\theta_{-1}\theta_2 + R^{--}{}^\alpha_\beta\theta_{-1}\theta_{-2} +$$
$$\mathrm{d}\theta_1\wedge{}^{(1)}B^\alpha_\beta + \mathrm{d}\theta_2\wedge{}^{(2)}B^\alpha_\beta + \mathrm{d}\theta_1\wedge\mathrm{d}\theta_2\,S^\alpha_\beta, \tag{37}$$

where

$$\begin{aligned}{}^{(1)}B^\alpha_\beta = \Big[&-\theta_2(\Gamma^{++}{}^\alpha_\beta - \Gamma^{-+}{}^\alpha_\beta + \mathrm{d}\Gamma_{(2)}{}^\alpha_\beta) - \theta_{-2}(\Gamma^{+-}{}^\alpha_\beta - \Gamma^{--}{}^\alpha_\beta + \mathrm{d}\tilde\Gamma_{(2)}{}^\alpha_\beta) +\\ &\theta_2\theta(0)(\Gamma^{++}{}^\gamma_\beta + \Gamma^{-+}{}^\gamma_\beta)\Gamma_{(2)}{}^\alpha_\gamma + \theta_{-2}\theta(0)(\Gamma^{+-}{}^\gamma_\beta + \Gamma^{--}{}^\gamma_\beta)\tilde\Gamma_{(2)}{}^\alpha_\gamma -\\ &\theta_2\theta(0)(\Gamma^{++}{}^\alpha_\gamma + \Gamma^{-+}{}^\alpha_\gamma)\Gamma_{(2)}{}^\gamma_\beta - \theta_{-2}\theta(0)(\Gamma^{+-}{}^\alpha_\gamma + \Gamma^{--}{}^\alpha_\gamma)\tilde\Gamma_{(2)}{}^\gamma_\beta\Big]_{\mathcal{B}_1},\end{aligned} \tag{38}$$

$$\begin{aligned}{}^{(2)}B^\alpha_\beta = \Big[&-\theta_1(\Gamma^{++}{}^\alpha_\beta - \Gamma^{+-}{}^\alpha_\beta + \mathrm{d}\Gamma_{(1)}{}^\alpha_\beta) - \theta_{-1}(\Gamma^{-+}{}^\alpha_\beta - \Gamma^{--}{}^\alpha_\beta + \mathrm{d}\tilde\Gamma_{(1)}{}^\alpha_\beta) +\\ &\theta_1\theta(0)(\Gamma^{++}{}^\gamma_\beta + \Gamma^{+-}{}^\gamma_\beta)\Gamma_{(1)}{}^\alpha_\gamma + \theta_{-1}\theta(0)(\Gamma^{-+}{}^\gamma_\beta + \Gamma^{--}{}^\gamma_\beta)\tilde\Gamma_{(1)}{}^\alpha_\gamma -\\ &\theta_1\theta(0)(\Gamma^{++}{}^\alpha_\gamma + \Gamma^{+-}{}^\alpha_\gamma)\Gamma_{(1)}{}^\gamma_\beta - \theta_{-1}\theta(0)(\Gamma^{-+}{}^\alpha_\gamma + \Gamma^{--}{}^\alpha_\gamma)\tilde\Gamma_{(1)}{}^\gamma_\beta\Big]_{\mathcal{B}_2},\end{aligned} \tag{39}$$

and

$$\begin{aligned}S^\alpha_\beta = \Big[&\Gamma_{(1)}{}^\alpha_\beta - \Gamma_{(2)}{}^\alpha_\beta - \left(\tilde\Gamma_{(1)}{}^\alpha_\beta - \tilde\Gamma_{(2)}{}^\alpha_\beta\right) +\\ &\theta(0)^2\left(\Gamma_{(2)}{}^\alpha_\gamma\Gamma_{(1)}{}^\gamma_\beta - \Gamma_{(1)}{}^\alpha_\gamma\Gamma_{(2)}{}^\gamma_\beta\right) + \theta(0)^2\left(\Gamma_{(2)}{}^\alpha_\gamma\tilde\Gamma_{(1)}{}^\gamma_\beta - \tilde\Gamma_{(1)}{}^\alpha_\gamma\Gamma_{(2)}{}^\gamma_\beta\right) +\\ &\theta(0)^2\left(\tilde\Gamma_{(2)}{}^\alpha_\gamma\Gamma_{(1)}{}^\gamma_\beta - \Gamma_{(1)}{}^\alpha_\gamma\tilde\Gamma_{(2)}{}^\gamma_\beta\right) + \theta(0)^2\left(\tilde\Gamma_{(2)}{}^\alpha_\gamma\tilde\Gamma_{(1)}{}^\gamma_\beta - \tilde\Gamma_{(1)}{}^\alpha_\gamma\tilde\Gamma_{(2)}{}^\gamma_\beta\right)\Big]_{\Sigma=\mathcal{B}_1\cap\mathcal{B}_2}.\end{aligned} \tag{40}$$

Further, if one wishes that eq. (37) describes only the intersection Σ one must impose on the brane 1-forms the following two additional restrictions:

$${}^{(a)}B^\alpha_\beta = {}^{(a)}b^\alpha_\beta\,\mathrm{d}\eta^{(a)}, \quad (a=1,2), \tag{41}$$

with ${}^{(a)}b^\alpha_\beta$ being the arbitrary functions containing θ's.

IV. DISCUSSION

Let us discuss now the features of singular $(V + E)$-submanifolds in details. Some of these features are drastically new with respect to both the physics of Kaluza-Klein type theories and that of the $(D-1)$-dimensional singular embeddings including $(D-2)$-branes as a special case. Then the two points of view on them, optimistic (constructive) and pessimistic (destructive), will be outlined. So, the features are:

(i) *The distinction of physics of singular manifolds from that of Kaluza-Klein dimensional reduction.*

It is well-known that nowadays the dimension of our visible Universe is regarded to be four due to many reasons, so higher-dimensional theories are obliged to eventually hide extra spacetime dimensions to fit the experimental data. For instance, the main style of thinking in the KK type models is to consider the *smooth* everywhere (except perhaps a finite number of points) high-D spacetime, decompose its values into the E and V (yet non-singular) parts, associating the former one with internal degrees of freedom, and then assume extra dimension compact with small size. Unlike this the singular manifold requires neither compactness nor hugeness of E-coordinates. It is entity which lives (moves and warps) inside a parent manifold (the latter becomes to be only C^0 in the vicinity of V), and has very own intrinsic geometry, matter, Standard Model (which is confined on V unlike gravitation), etc. However, the parent high-D manifold affects both the internal physics and external motion of the baby manifold, so as a consequence of this, there appears a number of distinctive features which are very specific for the singular submanifolds.

The higher-dimensional generation of internal degrees of freedom in the Universe V (in addition to the fibers of group spaces over the tangent bundle T_V) exists also in the KK type theories [18] as the phenomenon induced by E-metric g_{ab} and it is independent of whether we have the singular split $\Sigma = V \cup E$ or only the decomposition $T_\Sigma = T_V \oplus T_E$. Therefore, KK mechanism also works on a singular submanifold. However, the physics of latter is determined not only by confined matter (including the SM with its fiber bundles), by dynamics of V as a whole (also specific for singular submanifolds only), and by projected bulk Σ-gravity, but also by the high-D boundary effects including the anisotropy caused by the presence of multiple normals.

One may feel some vague analogy of this effect with the holography principle [21,22] which is also a boundary effect. However, at the present stage of the theory this connection yet seems to be too dim because the holography principle in its most radical form suggests that the information about volume processes is stored on the surface whereas the multi-normal embedding approach in initial form means the high-D mechanism of generation of internal degrees of freedom without introducing the fiber of internal symmetry groups.

(ii) *The (weak) violation of V-relativistic covariance and restricted structure of the parent manifold.*

The thorough look at the induced-gravity equations above and feature (i) reveals that the relativistic covariance is violated on the baby manifold V while preserved in the parent spacetime Σ. It can easily be seen that the effective stress-energy tensor will contain the terms which are not V-tensors. Generally speaking, the violation of relativity takes place for KK theories as well (because of the $(V+E)$-decomposition formulae are the same for both singular and non-singular submanifolds), but in that case the $(V + E)$-decomposition is at most than the mathematical relabeling of a high-D manifold, and hence the equations have no physical (induced-gravity) sense there. The relativity violation takes place also when considering $(D - 1)$-embeddings

(($D-2$)-branes) both in the junction and distributional approaches because it assumes the implicit separation of the extra dimension which is assumed to be orthogonal, i.e., similar to the gauss/synchronous coordinate. In the consistent theory of $(D-1)$-embeddings it is impossible to introduce the crucial definition of external curvature without this orthogonality. The assumption of orthogonality also means that if $D > 4$ we restrict ourselves to the spaces of a special "block-orthogonal" type: while the standard $(3+1)$ decomposition can be justified as the appropriately chosen coordinate system and hence implies no restrictions on the whole 4D manifold, we cannot say the same about 5D, 6D manifolds.

(iii) *The reformulation of the gravitational energy-momentum tensor problem.*
The embedded-world viewpoint can be applied to the old-standing problem of the conserved energy-momentum tensor for gravitational field (CGEMT) which is sometimes regarded as the main disadvantage of general relativity. Indeed, once we have imagined our Universe as the singular embedding inside the parent Meta-Universe there is no physical sense to require the conservation of the four-dimensional GEMT because the four-Universe explicitly becomes a gravitationally non-conservative system: gravity is not confined inside the four-manifold. The CGEMT problem is thus reduced to that of the higher-dimensional CGEMT. However, we can adjust the internal geometry and external dynamics of V in such a way that the high-D spacetime becomes flat or, at least, of constant curvature. Then the GEMT problem vanishes as well for the D-spacetime. The question is thus whether it is possible to do so that the perturbations of high-D metric caused by the matter on V and the external dynamics of V as a whole cancel each other out. Considering the emergent huge freedom of the as-a-whole external dynamics of singular submanifolds it seems to be possible, moreover, in a non-unique way.

(iv) *The parent manifold is not the manifold.*
The rigorous definition of the (differentiable) manifold require smoothness (including the smooth sewing of all the parts) and hence local diffeomorphicity to $\mathbf{R}^D$. However, in the vicinity of a singular submanifold the smoothness of the parent "manifold" breaks down [23], major definitions fail, and therefore still there is a question what is the physics "on the edge" [24].

After we have enumerated all the main objective peculiarities of the singular manifolds it is time to represent the subjective points of view on some of them.

• The *optimistic* (constructive) viewpoint suggests the following. The feature (i) does mean that the singular submanifold theories and brane-world models is new and promising mathematical tool and model of our Universe as part of the Meta-Universe. It provides us with opportunities to study the physical embeddings in the high-dimensional (even infinitely-dimensional) spaces. The feature (ii) is not the problem because the relativity holds for the whole spacetime Σ whereas the contributions violating the V-covariance can be regarded to give only the higher-order corrections to the induced Einstein equations ruling over the V(isible) Universe.

• If the hopes set upon the optimistic viewpoint will not be justified we should recall some disadvantages of the brane-world paradigm and seriously consider the *pessimistic* point of view. Indeed, looking back in time and comparing this paradigm to that of the Kaluza-Klein compactification we can see a number of defects. Apart from the problems mentioned above the serious one is that the non-uniqueness, which was inherent to the KK theories, is even more

amplified due to the appearance of the external dynamics of the baby manifold as a whole.[2] The researcher modeling physical reality by means of brane embeddings can obtain everything he wants and in several ways, and it does not seem to be a good sign because this decreases the foretelling ability of the theory. On the other hand, the experimental detectability of the high-D phenomena projected onto our Universe is a separate large problem [25,26].

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[2]Besides, we cannot completely replace the KK mechanism by the brane-world one because following the aforesaid the four-Universe can be singularly embedded only in five- and six-dimensional space so if we want to consider its embedding in 10D then other dimensions must be included but as a product space only.

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